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Antonio E. Noriega

Banco de México, Universidad de Guanajuato

Daniel Ventosa-Santaulària

Universidad de Guanajuato

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Spurious Regression and Econometric Trends*

Antonio E. Noriega[†]

Banco de México, Universidad de Guanajuato

Daniel Ventosa-Santaulària[‡]

Universidad de Guanajuato

Abstract

This paper analyses the asymptotic and finite sample implications of different types of non-stationary behavior among the dependent and explanatory variables in a linear spurious regression model. We study cases when the nonstationarity in the dependent and explanatory variables is deterministic as well as stochastic. In particular, we derive the order in probability of the t -statistic in a linear regression equation under a variety of empirically relevant data generation processes, and show that the spurious regression phenomenon is present in all cases considered, when at least one of the variables behaves in a nonstationary way. Simulation experiments confirm our asymptotic results.

Keywords: Spurious regression, trends, unit roots, trend stationarity, structural breaks.

JEL Classification: C12, C13, C22.

Resumen

Este documento analiza las implicaciones asintóticas y de muestras finitas de diferentes tipos de no-estacionariedad entre la variable dependiente y la explicativa en un modelo lineal de regresión espuria. Estudiamos casos de no estacionariedad en las variables del modelo tanto determinísticos como estocásticos. En particular, derivamos el orden en probabilidad del estadístico t en una ecuación de regresión lineal bajo una variedad de procesos generadores de datos empíricamente relevantes en economía, y mostramos que el fenómeno de regresión espuria está presente en todos los casos considerados, cuando al menos una de las variables se comporta de manera no estacionaria. Experimentos de simulación confirman los resultados asintóticos.

Palabras Clave: Regresión espuria, tendencias, raíces unitarias, estacionariedad en tendencia, cambios estructurales.

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[†] Dirección General de Investigación Económica. Email: anoriega@banxico.org.mx.

[‡] Escuela de Economía. Email: dventosa@yahoo.com.

1 Introduction

It has been documented in recent studies that the phenomenon of spurious regression is present under different forms of nonstationarity in the data generating process (*DGP*). In particular, when the variables y_t and x_t are nonstationary, independent of each other, ordinary least squares applied to the regression model

$$y_t = \alpha + \delta x_t + u_t$$

have the following implications: 1) the OLS estimator of δ ($\hat{\delta}$) does not converge to its true value of zero, and 2) the t -statistic for testing the null hypothesis $H_0 : \delta = 0$ ($t_{\hat{\delta}}$) diverges, thus indicating the presence of an asymptotic spurious relationship between y_t and x_t .

The rate at which $t_{\hat{\delta}}$ diverges depends on the type of nonstationarity present in the process generating y_t and x_t . In Phillips (1986), where a driftless random walk is assumed for both variables, the t -statistic is $O_p(T^{1/2})$. For the case of a random walk with drift, Entorf (1997) shows that $t_{\hat{\delta}}$ diverges at rate T . More recently, Kim, Lee and Newbold (2004) (KLN henceforth) show that the phenomenon of spurious regression is still present even when the nonstationarity in individual series is of a deterministic nature: they find that, under a linear trend stationary assumption for both variables, the t -statistic is $O_p(T^{3/2})$. Extending KLN's results, Noriega and Ventosa-Santaulària (2005) (NVS hereafter), show that adding breaks in the *DGP* still generates the phenomenon of spurious regression, but at a reduced divergence rate; i.e. $t_{\hat{\delta}}$ is $O(T^{1/2})$ under either single or multiple breaks in each variable. In all these works, the implicit assumption is that both variables share the same type of nonstationarity, either stochastic (Phillips, Entorf), or deterministic (KLN, NVS).³

Although the literature on this issue has grown considerably, there are still gaps, particularly when regressions involve variables with mixed types of trending mechanisms. The purpose of the present paper is to fill these gaps, using asymptotic and simulation arguments. Our results uncover the presence of spurious regressions under a wide variety of combinations of empirically relevant *DGPs*, not explored before in the literature. For instance, we consider regressions of a random walk with drift on a trend (with and without breaks)-stationary process (and vice versa). Section 2 discusses the *DGPs* considered. The asymptotic theory developed in Section 3 shows that the rate at which the phenomenon occurs is generally $T^{1/2}$, as predicted by Phillips (1998). However, for some combinations of trending mechanisms the divergence rate is higher. We also show that the spurious regression vanishes when one of the variables is stationary. Section 4 presents some simulation evidence for finite samples, while Section 5 concludes.

³Some related papers share this same feature: Marmol (1995, 1996, 1998), Cappuccio and Lubian (1997), Granger et. al. (1998) and Tsay and Chung (1999).

2 Trending mechanisms in the data generating process

In a simple regression equation, the nature of the trending mechanism in the dependent and explanatory variables is unknown a priori. This is mainly due to a lack of economic knowledge on trending mechanisms. We study the spurious regression phenomenon under eight different *DGPs*, widely used in applied work in economics.

We consider the following ordinary least squares regression model:

$$y_t = \hat{\alpha} + \hat{\delta}x_t + \hat{u}_t \quad (1)$$

used as a vehicle for testing the null hypothesis $H_0 : \delta = 0$. The following assumption summarizes the *DGPs* considered below for both the dependent and the explanatory variables in model (1).

ASSUMPTION. The *DGPs* for $z_t = y_t, x_t$ are as follows.

Case	Name*	Model
1.	I(0)	$z_t = \mu_z + u_{zt}$
2.	I(0)+ <i>br</i>	$z_t = \mu_z + \sum_{i=1}^{N_z} \theta_{iz} DU_{izt} + u_{zt}$
3.	TS	$z_t = \mu_z + \beta_z t + u_{zt}$
4.	TS+ <i>br</i>	$z_t = \mu_z + \sum_{i=1}^{N_z} \theta_{iz} DU_{izt} + \beta_z t + \sum_{i=1}^{M_z} \gamma_{iz} DT_{izt} + u_{zt}$
5.	I(1)	$\Delta z_t = u_{zt}$
6.	I(1)+ <i>dr</i>	$\Delta z_t = \mu_z + u_{zt}$
7.	I(1)+ <i>dr</i> + <i>br</i>	$\Delta z_t = \mu_z + \sum_{i=1}^{N_z} \theta_{iz} DU_{izt} + u_{zt}$
8.	I(2)	$\Delta^2 z_t = u_{zt}$

* TS, *br*, and *dr* stand for Trend-Stationary, breaks, and drift, respectively.

where u_{yt} and u_{xt} are independent innovations obeying the general level conditions of Assumption 1 in Phillips (1986), and DU_{izt} , DT_{izt} are dummy variables allowing changes in the trend's level and slope respectively, that is, $DU_{izt} = \mathbf{1}(t > T_{b_{iz}})$ and $DT_{izt} = (t - T_{b_{iz}})\mathbf{1}(t > T_{b_{iz}})$, where $\mathbf{1}(\cdot)$ is the indicator function, and $T_{b_{iz}}$ is the unknown date of the i^{th} break in z . We denote the break fraction as $\lambda_{iz} = (T_{b_{iz}}/T) \in (0, 1)$, where T is the sample size.

Note that cases 5, 6 and 7 can be written as

$$\begin{aligned} z_t &= z_0 + S_{zt} \\ z_t &= z_0 + \mu_z t + S_{zt} \\ z_t &= z_0 + \mu_z t + \sum_{i=1}^{M_z} \theta_{iz} DT_{izt} + S_{zt} \end{aligned}$$

where $S_{zt} = \sum_{i=1}^t u_{zi}$, $DT_{izt} = \sum_{i=1}^t DU_{izt}$, and z_0 is an initial condition.

Cases 1 and 2 have been used to model the behavior of (theoretically) mean stationary variables, such as real exchange rates, unemployment rates, the great

ratios (i.e. output-capital ratio, consumption-income ratio), and the current account. Examples of $I(0)$ and $I(0)$ variables with breaks have been presented in Perron and Vogelsang (1992), Wu (2000), and D’Adda and Scorcu (2003). Cases 3 to 8 are widely used to model growing variables, real and nominal, such as output, consumption, money, prices, etc. Macro variables have been described as $I(0)$ around a linear trend, $I(0)$ around a linear trend with structural breaks, and $I(1)$ in Perron (1992, 1997), Lumsdaine and Papell (1997), Mehl (2000), and Noriega and de Alba (2001). Combinations of case 8 with other cases are often behind the empirical modelling of nominal specifications expressed in terms of $I(2)$ (nominal) and $I(1)$ or $I(0)$ +breaks (real) variables. Economic models involving $I(2)$ variables include models of money demand relations, purchasing power parity, and inflation and the markup. Examples of variables described as $I(2)$ can be found in Juselius (1996, 1999), Haldrup (1998), Muscatelli and Spinelli (2000), Coenen and Vega (2001), and Nielsen (2002).

The *DGPs* include both deterministic and stochastic trending mechanisms, with 49 possible nonstationary combinations of them among the dependent and the explanatory variables, where case 1 is included as a benchmark.⁴

The spurious regression phenomenon has already been analyzed for a few combinations of *DGPs* in the Assumption. For instance, the case of both variables following a unit root (case 5) was studied by Granger and Newbold (1974) and Phillips (1986), and case 6 by Entorf (1997). The case (3) of a trend-stationary model for both variables was studied by KLN, while its extension to multiple breaks (case 4) by NVS. Mixtures of integrated processes were studied by Marmol (1995), who considers cases 5 and 8 (y follows a unit root, while x follows two unit roots, and vice versa). Many other combinations, however, have not been analyzed. Among them, combinations 3-6 and 4-6, which have practical importance, given the empirical relevance of structural breaks in the time series properties of many macro variables.

3 Asymptotics for spurious regressions

This section presents the asymptotic behavior of the t -statistic for testing the null hypothesis $H_0 : \delta = 0$ ($t_{\hat{\delta}}$) in model (1) when the dependent and explanatory variables are generated according to combinations of *DGPs* in the Assumption.

In the following Theorem, which collects the main results, a combination of *DGPs* is indicated by the pair $i - j$, ($i, j = 1, 2, \dots, 8$) indicating that y_t is generated by case i , while x_t by case j , both defined in the Assumption. Thus, for instance, the combination 8 - 5 corresponds to model (1) where y_t is $I(2)$ (case 8), while x_t is $I(1)$ (case 5).

⁴We do not consider the cases of $I(1)$ processes with long memory errors, and fractionally integrated processes, studied in Cappuccio and Lubian (1997), and Marmol (1998), respectively.

THEOREM. The order in probability of $t_{\hat{\delta}}$ in model (1) depends on the combination of DGPs for y_t and x_t in the Assumption, as follows:

- a) $t_{\hat{\delta}} = O_p(T^{-1/2})$ for combination of cases 1 – 1;
- b) $t_{\hat{\delta}} = O_p(1)$ for combination of cases 1 – i and i – 1, $i = 2, 3, \dots, 8$;
- c) $t_{\hat{\delta}} = O_p(T^{1/2})$ for combination of cases i – i , $i = 2, 4, 5, 7, 8$,
and i – j ; $i, j = 2, 3, \dots, 8$; $i \neq j$;
except for combinations 3 – 3, 3 – 6, 6 – 3 and 6 – 6;
- d) $t_{\hat{\delta}} = O_p(T)$ for combination of cases 3 – 6, 6 – 3 and 6 – 6;
- e) $t_{\hat{\delta}} = O_p(T^{3/2})$ for combination of cases 3 – 3.

PROOF. Results for combinations 3-3 and 1-3 come from KLN; for 5-5, 6-6, 8-8, and 5-8 results come from Phillips (1986), Entorf (1997), Marmol (1995), and Marmol (1996), respectively. The proof for the remaining 58 combinations was assisted by the software *Mathematica*, and, since the mechanics for obtaining the order in probability is the same for each combination of DGPs, we only present the procedure for a single combination, discussed in the Appendix at the end of the paper.

Part a) of the theorem shows that, as expected, when both variables are $I(0)$, the spurious regression phenomenon is not present, since the t -statistic collapses to zero (at rate \sqrt{T}). Part b) indicates that when one of the variables is $I(0)$ while the other follows any of the other nonstationary cases, the t -statistic neither collapses to zero, nor it diverges to infinite; instead, it converges (to a constant, or to a random variable, depending on the DGPs. See the small sample results of next section). For the majority of combination of cases, the t -statistic diverges (at rate \sqrt{T} or faster), indicating a spurious relationship among independent variables, as parts c)-e) show.

Table 1 summarizes the above findings. The symmetry of results imply that the order in probability does not depend on the type of nonstationarity among dependent and explanatory variables.

Table 1. Orders in probability of $t_{\hat{\delta}}$

DGP	1	2	3	4	5	6	7	8
1	$T^{-1/2}$	1	1	1	1	1	1	1
2		$T^{1/2}$	$T^{1/2}$	$T^{1/2}$	$T^{1/2}$	$T^{1/2}$	$T^{1/2}$	$T^{1/2}$
3			$T^{3/2}$	$T^{1/2}$	$T^{1/2}$	T	$T^{1/2}$	$T^{1/2}$
4				$T^{1/2}$	$T^{1/2}$	$T^{1/2}$	$T^{1/2}$	$T^{1/2}$
5					$T^{1/2}$	$T^{1/2}$	$T^{1/2}$	$T^{1/2}$
6						T	$T^{1/2}$	$T^{1/2}$
7							$T^{1/2}$	$T^{1/2}$
8								$T^{1/2}$

Therefore, when two independent random variables follow any of the nonstationary combinations considered in the Assumption, OLS inference will indicate, asymptotically, a significant (spurious) relationship among them.

The representation theory developed by Phillips (1998) shows that a trending stochastic (or deterministic) process can be represented in various ways. In particular, it can be written as an infinite linear combination of trending deterministic (stochastic) functions with random coefficients. In such an asymptotic environment, he shows that the regression t -ratios of the fitted coefficients diverge at rate $O_p(T^{1/2})$. Results from the theorem above indicate that relatively simple nonstationary time series models correctly indicate the presence of the limiting representation.⁵

4 Experimental results

We computed rejection rates of the t -statistic for testing the null hypothesis $H_0 : \delta = 0$, in model (1), using a 1.96 critical value (5% level) for a standard normal distribution. In order to assess the usefulness for a finite sample of the asymptotic results presented in the theorem, rejection rates were based on simulated data, for samples of size $T = 25, 50, 100, 250, 500, 1,000$, and $10,000$, under various combinations of DGP s in the Assumption.⁶ In all experiments, the number of replications is 10,000.

Table 2. Rejection Rates for $t_{\hat{\delta}}$; the case of two breaks

T	Combinations of cases (DGP s) in the Assumption											
	1-7	2-2	2-4	2-6	3-4	3-6	4-4	4-6	4-7	4-8	6-8	7-8
25	.06	.06	.06	.07	.14	.25	.32	.61	.66	.65	.93	.94
50	.06	.06	.06	.06	.89	.95	.99	.99	.99	.94	.95	.96
100	.05	.05	.06	.06	1.0	.99	1.0	1.0	1.0	.97	.97	.97
250	.05	.05	.07	.07	1.0	1.0	1.0	1.0	1.0	.98	.98	.98
500	.06	.05	.10	.10	1.0	1.0	1.0	1.0	1.0	.99	.99	.99
1000	.05	.05	.14	.15	1.0	1.0	1.0	1.0	1.0	.99	.99	.99
10000	.05	.05	.80	.79	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0

The values of the parameters in the DGP s are as follows: $\sigma_z = 1$, $\phi_z = 0$, $\mu_x = 0.4$, $\mu_y = 0.7$, $\beta_x = 0.07$, $\beta_y = 0.04$, $\theta_{xi} = 0.07$, $\theta_{yi} = 0.04$, $\gamma_{xi} = 0.02$, $\gamma_{yi} = 0.04$, for $i = 1, \dots, M_z$, $M_z = 2$, for $z = x, y$. Breaks in x (y) occur at 20% (40%) and 70% (80%) of total data length.

⁵A few cases do not conform with the general level results of Phillips (1998) in terms of divergence rates. These are the cases 3-3, 3-6, and 6-6 (the first one studied in KLN and the last one in Entorf (1997)). In all three cases the divergence rates are higher.

⁶The experimental results in this section are limited, and only serve as a guide on the finite sample behaviour of some particular cases. The values of the parameters were inspired on real data from Perron and Zhu (1995), comprising historical real percapita GDP series for industrialized economies.

Table 3. Rejection Rates for $t_{\hat{\delta}}$; the case of four breaks

T	Combinations of cases (DGP s) in the Assumption											
	1-7	2-2	2-4	2-6	3-4	3-6	4-4	4-6	4-7	4-8	6-8	7-8
25	.06	.06	.06	.07	.18	.25	.81	.89	.97	.90	.93	.94
50	.06	.06	.07	.07	.94	.94	1.0	.99	.99	.96	.96	.96
100	.05	.05	.09	.09	1.0	1.0	1.0	1.0	1.0	.97	.97	.97
250	.05	.05	.14	.14	1.0	1.0	1.0	1.0	1.0	.98	.98	.98
500	.05	.05	.23	.23	1.0	1.0	1.0	1.0	1.0	.99	.99	.99
1000	.05	.05	.41	.41	1.0	1.0	1.0	1.0	1.0	.99	.99	.99
10000	.05	.08	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0

Experimental design as above, but with $M_z = 4$, for $z = x, y$. Breaks in x (y) occur at 15% (20%), 30% (35%), 50% (55%) and 70% (80%) of total data length.

Tables 2 and 3 present rejection rates under 12 different combinations of DGP s in the Assumption. In Table 2, the cases where breaks are considered (all but 3-6 and 6-8) include 2 breaks, while Table 3 presents results when 4 breaks are allowed. The column labeled 1-7 in both tables presents the finite sample counterpart of the $O_p(1)$ result in part *b*) of the theorem: the t -statistic does not diverge, indicating that the spurious regression phenomenon is not a problem in finite samples. The tables also show that a nonsense regression is not likely in small samples ($25 \leq T \leq 250$) when one of the variables is generated by an $I(0)$ process with structural breaks (combinations of cases 2-2, 2-4, and 2.6), as long as there are not too many breaks.⁷ For the rest of cases, the asymptotic nonsense relationship reported in the theorem is also detected in our simulation experiments: the spurious regression phenomenon is present even for samples as small as 25. In comparing results from Tables 2 and 3, it can be noted that a nonsense regression is more likely when the number of structural breaks increases in the DGP .

5 Conclusions

This paper has presented an asymptotic and experimental analysis of the spurious regression phenomenon under a wide variety of empirically relevant data generating processes in a simple regression model. It fills many gaps left open by previous research in the area. In particular, it shows that the t -statistic for testing a linear relationship among independent time series diverges, if both variables show a nonstationary behavior, due to either stochastic (unit roots) or deterministic (linear trends and/or structural breaks) factors.

Our results particularize Phillips' (1998) general results to empirically useful models, by showing that the phenomenon of spurious regression is present for time series with relatively simple trending mechanisms. This phenomenon

⁷For case 2-2 the simulation results seem not to detect a spurious regression even with 10,000 observations. For this particular case, the finite sample results converge very slowly to the asymptotic one. When the sample size is increased to 100,000 (400,000), we obtained rejection rates of 7.2% (11.8%) for $M_z = 2$, and 39.9% (91.9%) for $M_z = 4$.

depends on the commonality of trends and/or breaks in both dependent and explanatory variables. If the nonstationary behavior (stochastic or deterministic) is present in only one of the variables, however, the spurious regression vanishes. Our simulation experiments reveal that a spurious regression is not exclusively an asymptotic phenomenon, it will also be present in finite samples for the majority of *DGPs* considered.

Given that trending mechanisms as the ones analyzed here are a common feature of the long-run behaviour in many macroeconomic time series, the main conclusion points to a warning regarding inferences drawn from OLS regression analysis: the probability of finding a nonsense correlation among independent series will not only be high in finite samples, but it will grow with the sample size.

6 Appendix

We present a guide on how to obtain the order in probability of one combination of *DGPs*, namely, the combination 1-7, for which

$$y_t = \mu_y + u_{yt}$$

$$x_t = x_0 + \mu_x t + \sum_{i=1}^{M_x} \theta_{ix} DT_{ixt} + S_{xt}$$

The orders in probability for the rest of cases follow the same steps. The proofs were assisted by the software *Mathematica 4.1*. The corresponding codes for all combinations of *DGPs* are available at www.ventosa-santaularia.com/NVS_06a.zip.

Below, we describe the steps involved in the computerized calculations.

Write the regression model $y_t = \alpha + \delta x_t + u_t$ in matrix form: $y = X\beta + u$. The vector of *OLS* estimators is $\hat{\beta} = (\hat{\alpha} \quad \hat{\delta})' = (X'X)^{-1}X'y$, and the *t*-statistic of interest $t_{\hat{\delta}} = \hat{\delta} \left[\hat{\sigma}_u^2 (X'X)^{-1}_{22} \right]^{-1/2}$, where $(X'X)^{-1}_{22}$ is the 2^{nd} diagonal element of $(X'X)^{-1}$ and $\hat{\sigma}_u^2 = T^{-1} \sum_{t=1}^T \hat{u}_t^2 = T^{-1} \sum_{t=1}^T \left(y_t - \hat{\alpha} - \hat{\delta} x_t \right)^2$. $t_{\hat{\delta}}$ is a function of the following objects:

$$\sum_{t=1}^T y_t = \mu_y T + \Sigma_{uy} T^{1/2}$$

$$\sum_{t=1}^T y_t^2 = (\mu_y^2 + \Sigma_{u2y}) T + 2\mu_y \Sigma_{uy} T^{1/2}$$

$$\sum_{t=1}^T x_t = \frac{1}{2} \left[\mu_x + \sum_{i=1}^{M_x} \theta_i (1 - \lambda_i)^2 \right] T^2 + \Sigma_{sx} T^{3/2} + \left[x_0 + \frac{1}{2} \left(\mu_x + \sum_{i=1}^{M_x} \theta_i (1 - \lambda_i) \right) \right] T$$

$$\sum_{t=1}^T x_t^2 = \left[\frac{1}{3} \mu_x^2 + \lambda^+ + \frac{1}{3} \mu_x \sum_{i=1}^{M_x} \theta_i (1 - \lambda_i)^2 (\lambda_i + 2) \right] T^3 + 2(\mu_x \Sigma_{tsx} + \Sigma_{ts1xi}) T^{5/2} + O_p(T^2)$$

$$\sum_{t=1}^T y_t x_t = \left[\frac{1}{2} \mu_y (\mu_x + \sum_{i=1}^{M_x} \theta_i (1 - \lambda_i)^2) \right] T^2 + O_p(T^{3/2})$$

with

$$\Sigma_{uy} = T^{-1/2} \sum_{t=1}^T u_{yt}$$

$$\begin{aligned}
\Sigma_{u2y} &= T^{-1} \sum_{t=1}^T u_{yt}^2 \\
\Sigma_{sx} &= T^{-3/2} \sum_{t=1}^T S_{xt} \\
\Sigma_{tsx} &= T^{-5/2} \sum_{t=1}^T t S_{xt} \\
\Sigma_{ts1xi} &= T^{-5/2} \sum_{i=1}^{M_x} \theta_i \left(\sum_{t=T_{b_i}+1}^T t S_{xt} - \lambda_i \sum_{t=T_{b_i}+1}^T S_{xt} \right) \\
\lambda^+ &= \frac{1}{3} \sum_{i=1}^{M_x} \theta_i^2 (1 - \lambda_i)^2 + \sum_{i=1}^{M_x-1} \sum_{j=i+1}^{M_x} \theta_i \theta_j \left[\frac{2}{3} (1 - \lambda_{u(i,j)})^3 + \lambda_{d(i,j)} (1 - \lambda_{u(i,j)})^2 \right] \\
\lambda_{u(i,j)} &= \max(\lambda_i, \lambda_j), \quad i, j = 1, 2, \dots, M_x \\
\lambda_{l(i,j)} &= \min(\lambda_i, \lambda_j) \\
\lambda_{d(i,j)} &= \lambda_{u(i,j)} - \lambda_{l(i,j)} \\
\text{where (see for instance Phillips (1986)),} \\
\Sigma_{uy} &\Rightarrow \sigma_y W_y(1) \\
\Sigma_{u2y} &\Rightarrow \sigma_y^2 \\
\Sigma_{sx} &\Rightarrow \sigma_x \int_0^1 W_x(r) dr \\
\Sigma_{tsx} &\Rightarrow \sigma_x \int_0^1 r W_x(r) dr \\
\Sigma_{ts1xi} &\Rightarrow \sigma_x \sum_{i=1}^{M_x} \theta_i \int_{\lambda_i}^1 (r - \lambda_i) W_x(r) dr,
\end{aligned}$$

\Rightarrow signifies convergence in distribution, and $W_z(r)$, $z = y, x$ is the standard Wiener process on $r \in [0, 1]$.

Using these expressions, *Mathematica* computes the limiting distribution of the parameter vector and the rest of the elements of $t_{\hat{\delta}}$ by factoring out the relevant expressions in powers of the sample size. In this way, the orders in probability can be determined, and the limiting expression obtained, by retaining only the asymptotically relevant terms, upon a suitable normalization. From *Mathematica*'s output it can be deduced that, for the case at hand,

$$T^{3/2} \hat{\delta} \left[\hat{\sigma}_u^2 T^3 (X'X)_{22}^{-1} \right]^{-1/2} = \hat{\delta} \left[\hat{\sigma}_u^2 (X'X)_{22}^{-1} \right]^{-1/2} = O_p(1),$$

as reported in the Theorem.

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